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Note

Critical groups for homeomorphism classes of graphs

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Abstract

In this paper the weighted fundamental circuits intersection matrix of an edge-labeled graph is introduced for computing the critical groups for homeomorphism classes of graphs. As an application, it is proved that for any given finite connected simple graph there is a homeomorphic graph with cyclic critical group.

Keywords: Critical group; Laplacian matrix; Homeomorphism class; Cyclic group

1. Introduction

Let $G = (V, E)$ be a finite connected simple graph and $n = |V|$. The $n \times n$ Laplacian matrix $L(G)$ for this graph G is defined by

$$L(G)_{u,v} = \begin{cases} \deg_G(u) & \text{if } u = v \\ -1 & \text{if } \{u, v\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

In the following we always consider an $n \times n$ integer matrix as a map from $\mathbb{Z}^n$ to itself. Since G is connected, the kernel of $L(G)$ is the cyclic group generated by the vector $(1, 1, \ldots, 1)^t$ in $\mathbb{Z}^n$ where the superscript t denotes the transpose. The cokernel of $L(G)$ has the form $\mathbb{Z}^n/\mathrm{Im}\, L(G) \cong \mathbb{Z} \oplus K(G)$, where $K(G)$ is defined to be the *critical group*. It follows from Kirchhoff's Matrix–Tree Theorem (see e.g. [3]) that the order $|K(G)|$ is $\kappa(G)$, the number of spanning trees in G. There exist many results relating the group structure of $K(G)$ to the graphical structure of G (cf. [1,2,4,5,7,8,10]).

Two integer matrices A and B are *equivalent* if there exist integer matrices P and Q of determinant ± 1 such that $PAQ = B$. Given a square integer matrix A, its *Smith normal form* is the unique *equivalent* diagonal matrix $S(A) = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ whose entries s_i are nonnegative and s_i divides s_{i+1}. The s_i and $d_i = \prod_{1 \le j \le i} s_j$ $(1 \le i \le n)$ are known as the *invariant factors* and *determinantal divisors* of A respectively (cf. [11]).

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The structure of the critical group is closely related to the Laplacian matrix: if the Smith normal form of $L(G)$ is $\mathrm{diag}[s_1, s_2, \ldots, s_n]$, $K(G)$ is the torsion subgroup of $\mathbb{Z}_{s_1} \times \mathbb{Z}_{s_2} \times \cdots \times \mathbb{Z}_{s_n}$. Here $\mathbb{Z}_h$ denotes the cyclic group $\mathbb{Z}/h\mathbb{Z}$ for any nonnegative integer h.

In [12,13], Read and Whitehead studied the chromatic and Tutte polynomials for homeomorphism classes of graphs respectively. In this paper, we study critical groups for homeomorphism classes of graphs through the weighted fundamental circuits intersection matrix (see Section 2.2).

The main result is as follows.

Theorem 1. *For any connected graph G with n vertices and m edges, the critical group of the graph obtained by a suitable subdivision of at most $m - n$ edges of G is cyclic.*

2. Weighted fundamental circuits intersection matrix

2.1. Fundamental circuits intersection matrix

Let $G = (V, E)$ be a finite connected simple graph, $n = |V|$ and $m = |E|$. Denote by $\overrightarrow{G} = (V, \overrightarrow{E})$ the unique symmetric directed graph corresponding to G, i.e., oriented edges $uv, vu \in \overrightarrow{E}$ if and only if $\{u, v\} \in E$. For $e \in \overrightarrow{E}$, we denote by $o(e)$ and $t(e)$ the origin and the terminus of e, respectively. The inverse edge of e is denoted by $\bar{e}$. An orientation of $\overrightarrow{G}$ is given by a subset E^0 of E such that $\overrightarrow{E} = E^0 \cup \bar{E^0}$, where $\bar{E^0} = \{\bar{e} | e \in E^0\}$. A path in G is a sequence $c = (e_1, e_2, \ldots, e_l)$ of oriented edges such that $t(e_i) = o(e_{i+1})$ for $i = 1, 2, \ldots, l-1$. We write $o(c) = o(e_1)$ and $t(c) = t(e_l)$.

Let $\langle E \rangle$ be the real vector space generated by elements of $\overrightarrow{E}$ with relations $\bar{e} = -e (e \in \overrightarrow{E})$. We put

$$e \cdot e' = \begin{cases} 1 & \text{if } e = e' \\ -1 & \text{if } \bar{e} = e' \\ 0 & \text{otherwise} \end{cases}$$

for $e, e' \in E$. Then we extend $\cdot$ bilinearly to $\langle E \rangle$ to obtain an inner product on the vector space $\langle E \rangle$.

Take a spanning tree $T = (V_T, E_T)$ in G. The tree T contains all of the n vertices of G and the number of edges in T is $n - 1$. Let $t = m - n + 1$. Choose one orientation $E^0 = \{e_1, e_2, \ldots, e_t, e_{t+1}, e_{t+2}, \ldots, e_m\}$ such that $e_i \notin \overrightarrow{E_T} (i \leqslant t)$ and $e_i \in \overrightarrow{E_T} (i > t)$. For each $1 \le i \le t$, take the minimal path p_i in $\overrightarrow{T}$ such that $o(p_i) = t(e_i)$ and $t(p_i) = o(e_i)$, and put $c_i = e_i p_i$. Then each c_i is a circuit, i.e., it has no self-intersection.

Write $V = \{v_1, v_2, \ldots, v_n\}$. The incidence matrix is the $n \times m$ matrix $B = (b_{ij})$ with elements $b_{ij} \in \mathbb{Z}$ defined by

$$\partial e_j := t(e_j) - o(e_j) = \sum_{i=1}^{n} b_{ij} v_i$$

for $e_j \in E^0 (j = 1, 2, \ldots, m)$. Then $L(G) = BB^t$. For any $1 \le i \le t$, let

$$s_i = e_i + \sum_{j=1}^{n-1} f_{ij} e_{t+j},$$

where

$$f_{ij} = \begin{cases} 1 & \text{if } e_{t+j} \in p_i \\ -1 & \text{if } \overline{e_{t+j}} \in p_i \\ 0 & \text{otherwise.} \end{cases}$$

Call $F = (f_{jl})$ the fundamental circuits matrix. Define the *fundamental circuits intersection matrix* $C = (c_{ij})$ by $c_{ij} = s_i \cdot s_j$. Noting that $f_{ij} = s_i \cdot e_{t+j}$, we have

$$C = I + FF^t.$$

Almost all of the definitions above are from [9]. The following simple but useful lemma, which is the starting point of the paper, is also motivated by Lemma 3.4 in [9].

Lemma 2. *The critical group of a graph is isomorphic to the cokernel of the fundamental circuits intersection matrix of the graph.*

Proof. Take a spanning tree T and fix an orientation E^0 of G as above. The reduced incidence matrix $\widetilde{B}$ obtained from B by omitting the last row is of the form

$$\widetilde{B} = (B_1, B_2) = \left(\begin{array}{ccc|ccc} b_{11} & \cdots & b_{1,t} & b_{1,t+1} & \cdots & b_{1,m} \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{n-1,1} & \cdots & b_{n-1,t} & b_{n-1,t+1} & \cdots & b_{n-1,m} \end{array}\right),$$

where $B_2 \in GL_{n-1}(\mathbb{Z})$ (cf. [6]).

From

$$0 = \partial s_i = \partial e_i + \sum_{j=1}^{n-1} f_{ij} \partial e_{t+j}$$

it follows that $B_1 + B_2 F^t = 0$ and therefore $F = (-B_2^{-1} B_1)^t$. We have $\widetilde{B} = B_2(-F^t, I)$, and thus

$$\widetilde{B}\widetilde{B}^t = B_2(-F^t, I)(-F^t, I)^t B_2^t = B_2(F^t F + I) B_2^t.$$

Note that

$$\begin{pmatrix} I & -F \\ 0 & I \end{pmatrix} \begin{pmatrix} I + FF^t & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ F^t & I \end{pmatrix} = \begin{pmatrix} I & -F \\ F^t & I \end{pmatrix},$$

and

$$\begin{pmatrix} I & 0 \\ F^t & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I + F^t F \end{pmatrix} \begin{pmatrix} I & -F \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & -F \\ F^t & I \end{pmatrix}.$$

Thus $K(G) = \operatorname{Coker}(\widetilde{B}\widetilde{B}^t) = \operatorname{Coker}(I + F^t F)$ is isomorphic to $\operatorname{Coker}(I + FF^t) = \operatorname{Coker} C$. The proof is complete. □

2.2. Weighted fundamental circuits intersection matrix

Consider a graph $G = (V, E)$ with an edge-labeling function $\omega : \overrightarrow{E} \to \mathbb{Z}^+$ such that $\omega(\bar{e}) = \omega(e)$. This edge-labeled graph will represent the graph G^ω obtained from G by replacing each edge e with a chain of $\omega(e)$ edges in series. If $\widetilde{E} = \{e \in E | \omega(e) > 1\}$, then G^ω is obtained by some subdivision of $\widetilde{E}$. We say that two graphs L_1 and L_2 are in the same homeomorphism class if there exists a graph L that is a subdivision of both L_1 and L_2. If a graph G is edge-labeled, then we put

$$e \odot e' = \begin{cases} \omega(e) & \text{if } e = e' \\ -\omega(e) & \text{if } \bar{e} = e' \\ 0 & \text{otherwise} \end{cases}$$

for $e, e' \in \overrightarrow{E}$. We can extend $\odot$ bilinearly to $\langle E \rangle$ to obtain an inner product on the vector space $\langle E \rangle$. Let c_i and s_i be as above. We call the $t \times t$ matrix $C^\omega = (s_i \odot s_j)$ the *weighted fundamental circuits intersection matrix* of the graph G with the edge-labeling function ω.

Proposition 3. *The critical group of an edge-labeled graph is isomorphic to the cokernel of the weighted fundamental circuits intersection matrix of the labeled graph.*

Proof. Take a spanning tree T and fix an orientation E^0 of G as in the subsection above. Suitably choosing a spanning tree T and an orientation E^0 of G^ω, the fundamental circuits intersection matrix of G^ω is just the weighted fundamental circuits intersection matrix of the graph G with the edge-labeling function ω. So the result follows from Lemma 2. The proof is complete. □

3. Proof of Theorem 1

To prove Theorem 1 we need the following lemma.

Lemma 4. *For a positive definite matrix $C = (\alpha_{ij}) \in M_t(\mathbb{Z})$, the Smith normal form of $C^* = \operatorname{diag}(0, x_2, \ldots, x_t) + C$ can be of the form* $\operatorname{diag}(1, 1, \ldots, 1, d)$ *by suitable choice of nonnegative integers $x_2, \ldots, x_t$.*

Proof. We perform induction on t. The result is trivial for $t = 1$. Assume that we have got the result for $t \leq k$. When $t = k + 1$, note that the upper left $k \times k$ matrix is positive definite. By induction, there exist unimodular matrices P and Q, and $d' \in \mathbb{Z}$, such that

$$\bar{C} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} C^* \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & & \alpha_{1,k+1} \\ & \ddots & & & \vdots \\ & & 1 & & \alpha_{k-1,k+1} \\ & & & d' & \alpha_{k,k+1} \\ \alpha_{k+1,1} & \cdots & \alpha_{k-1,k+1} & \alpha_{k,k+1} & x_{k+1} + \alpha_{k+1,k+1} \end{pmatrix}$$

for some nonnegative integers $x_2, \ldots, x_k$. Since C is positive definite, so is C^*. Thus we have $d' \in \mathbb{Z} \backslash \{0\}$. Now it suffices to find two $k \times k$ minor matrices of $\bar{C}$ whose determinants are relatively coprime for some nonnegative integer x_{k+1}. Denote by $\bar{C}(i)$ the determinant of the principal minor matrix obtained by deleting the i-th column and the i-th row of $\bar{C}$. Then $\bar{C}(k) = x_{k+1} + \alpha_{k+1,k+1} + \beta$ for some $\beta \in \mathbb{Z}$ and $\bar{C}(k+1) = d' \in \mathbb{Z} \backslash \{0\}$. We can choose a nonnegative integer x_{k+1} such that the two minors $\bar{C}(k)$ and $\bar{C}(k+1)$ are relatively coprime. The proof is complete. □

Now we can complete the proof of Theorem 1.

Proof. Take a spanning tree T and fix an orientation E^0 of G as above. Suppose that C is the fundamental circuits intersection matrix of G. It is well known that the reduced Laplacian matrix of a connected graph is positive definite. Choose $x_2, x_3, \ldots, x_t$ as in Lemma 4, and subdivide $e_2, e_3, \ldots, e_t$ as paths of length $x_2 + 1, x_3 + 1, \ldots, x_t + 1$ respectively. Then we get an edge-labeled graph G and the weighted fundamental circuits intersection matrix of G is $C^\omega = (0, x_2, \ldots, x_t) + C$. By Proposition 3, the critical group of G^ω is cyclic. The proof is complete. □

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